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REDUCTION OF ADDITIVE COLORED NOISE USING COUPLED DYNAMICS

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We study the effect of additive colored noise on the evolution of maps and demonstrate that the deviations caused by such noise can be reduced using coupled dynamics. We consider both Ornstein-Uhlenbeck process as well as $1/f^\alpha$ noise in our numerical simulations. We observe that though the variance of deviations caused by noise depend on the correlations in the noise, under optimal coupling strength, it decreases by a factor equal to the number of coupled elements in the array as compared to variance of deviations in a single isolated map. This reduction in noise levels occurs in chaotic as well as periodic regime of the maps. Lastly, we examine the effect of colored noise in chaos computing and find that coupling the chaos computing elements enhances the robustness of chaos computing.

Keywords: Coupled map lattice, Colored noise, $1/f^\alpha$ noise, Ornstein-Uhlenbeck process, Chaos Computing

1. Introduction

Dynamical systems have been used for modeling many natural and engineering systems [Grebogi & Ott , 1993; Sinha & Ditto, 1998; Kia *et al.*, 2011; Ercsey-Ravasz & Toroczkai, 2011; Sumi *et al.*, 2014; Lindner *et al.*, 2015] and their interaction with intrinsic as well as environmental noise has been extensively studied. These studies have unveiled a variety of interesting phenomenon like stochastic resonance [Gammaitoni *et al.*, 1998; Lindner *et al.*, 1995; Kohar & Sinha , 2012; Choudhary *et al.*, 2014] in which noise aids the functioning of such systems but at the same time there are many other applications where noise deteriorates the performance and thus suppression of noise is of crucial importance in such systems. Recently, it was shown that coupling nonlinear dynamical systems reduces the effects of white Gaussian noise [Kia *et al.* , 2015]. Specifically, it was shown that noise from different lattice nodes in a coupled map lattice diffuses across the lattice and lowers the noise level in the individual nodes. This phenomenon was also utilized to enhance the robustness of chaos computing systems in the presence of local noise [Kia *et al.*, 2014].

White noise was used in these studies to demonstrate the reduction of noise using coupled dynamics. The assumption of white noise works well in situations where the characteristic time scale of the system is much larger than that of random perturbations due to noise. Some studies have demonstrated that presence of color in the noise can result in deviations from the behavior observed for white noise [Hanggi & Jung , 1995; Nozaki *et al.* , 1999; Anishchenko & Neiman , 1992]. Thus it becomes imperative that the effect of colored noise is also studied and the concepts demonstrated for white noise are also tested for colored noise to understand the effects of correlations in the noise [Garca-Ojalvo & Sancho , 1999; Hanggi & Jung , 1995]. Two common approaches of modeling the colored noise include the exponentially correlated Gaussian noise modeled by Ornstein-Uhlenbeck (OU) process as in Gardiner [1985] and the $1/f^{\alpha}$ noise as in Weissman [1988] where α denotes the scaling exponent of the power spectrum of noise and thus defines the color of noise.

In this paper, we study the effect of colored noise on evolution of maps and its implications for chaos computing. We extend the results obtained for white noise in [Kia *et al.* , 2015; Kia *et al.*, 2014] to colored noise and demonstrate that they are valid for additive Gaussian colored noise also. Further, we investigate how the deviations of map are affected by the correlations in the noise. We also evaluate the robustness of chaos computing in the presence of colored noise and show that the phenomenon of reduction of colored noise using coupled dynamics can be used in chaos computing [Kia *et al.*, 2014] and coupling the chaotic computing elements reduces the errors due to additive Gaussian colored noise leading to more robust chaos computing.

In the next section, we revisit the theoretical framework on how coupling the dynamical elements results in reduction of noise. In Section III, we present our numerical results for additive Gaussian colored noise modeled by OU process in coupled sine and quadratic maps. In Section IV, we present the concept of chaos computing and demonstrate that coupled chaotic elements can function robustly at noise strengths higher than those of uncoupled ones. In next section, we discuss the effects of $1/f^{\alpha}$ noise and conclude with a summary of important results in the last section.

2. Theoretical Framework

Let us consider a generalized globally coupled map lattice (CML) [Kaneko, 1992] defined as

$$x_n^{i+1} = (1-\epsilon) f[x_n^i] + \frac{\epsilon}{(N-1)} \sum_{m \neq n} f[x_m^i], \qquad (1)$$

where x_n^i is the state of the n^{th} node at the i^{th} iteration, N is the total number of nodes, ϵ is the coupling strength and $f[x]$ is an one dimensional map. For $\epsilon = 0$, Eq 1 reduces to uncoupled dynamics at each node given by

$$x^{i+1} = f[x^i]. \qquad (2)$$

Now we study how local noise affects the evolution of this CML and whether coupling can reduce the effects of this noise. We consider that the CML evolves in the presence of a statistically independent but identically distributed Gaussian colored noise with zero mean and standard deviation σ. Thus the evolution of CML is modified as

$$x_n^{i+1} = (1-\epsilon) f[x_n^i] + \frac{\epsilon}{(N-1)} \sum_{m \neq n} f[x_m^i] + \sigma \delta_n^{i+1}. \qquad (3)$$

Next we analyze how noise affects the state of the nodes on successive iterations. To start with, we initialize all the nodes of the CML with the same initial condition. Due to presence of noise, the initial conditions of the CML are modified as $x_n^0 \to x_n^0 + \sigma\delta_n^0 = x^0 + \sigma\delta_n^0$. Notice that though the noise is identically distributed, it is local and statistically independent and hence the instantaneous level of noise at each node is different. The state of n^{th} node at first iteration is given by

$$x_n^1 = (1-\epsilon) f[x^0 + \sigma\delta_n^0] + \frac{\epsilon}{(N-1)} \sum_{m \neq n} f[x^0 + \sigma\delta_m^0] + \sigma\delta_n^1. \qquad (4)$$

We assume that the instantaneous noise levels are low such that in the Taylor expansion of $f(x+\sigma\delta_n^i)$ second and higher order terms are negligible. Thus linearizing the above equation around x^0 with the assumption that $\lambda_1 \neq 0$ where $\lambda_1 = df(x^0)/dx$ and neglecting second and higher order terms, we obtain

$$x_n^1 \approx f[x^0] + \lambda_1 \left((1-\epsilon)\sigma\delta_n^0 + \frac{\epsilon}{(N-1)} \sum_{m \neq n} \sigma\delta_m^0 \right) + \sigma\delta_n^1. \qquad (5)$$

The first term in Eq 5 is the noise free evolution of a single map. The second term consists of evolution of noise terms and the last term is the noise added at the current iteration. In the second term, the noise terms are independent and identically distributed and so the total variance of second term will be equal to sum of variances of individual terms. Thus the variance of deviations of the CML at first iteration is

$$\begin{aligned} \sigma_{CML1}^2 &= \lambda_1^2 \left((1-\epsilon)^2\sigma^2 + \sum_{m \neq n} \frac{\epsilon^2}{(N-1)^2}\sigma^2 \right) \\ &= (\sigma\lambda_1)^2 \left((1-\epsilon)^2 + \frac{\epsilon^2}{N-1} \right). \end{aligned} \qquad (6)$$

For the uncoupled case ($\epsilon = 0$), the variance of deviations in a single map evolving in the presence of noise reduces to

$$\sigma^2_{SML1} = (\sigma\lambda_1)^2 = \left((1-\epsilon)^2 + \frac{\epsilon^2}{N-1}\right)^{-1}\sigma^2_{CML1}. \tag{7}$$

To compare the deviations due to noise in the CML with those of an isolated map, we define the ratio of variances of deviations in the CML (σ^2_{CML}) and single map (σ^2_{SML}) to be the noise robustness, R.

$$R = \frac{\sigma^2_{SML}}{\sigma^2_{CML}}. \tag{8}$$

Using Eq 8, noise robustness after first iteration is

$$R = \frac{\sigma^2_{SML1}}{\sigma^2_{CML1}} = \frac{1}{((1-\epsilon)^2 + \epsilon^2/(N-1))}. \tag{9}$$

Maximizing R in Eq 9 gives $\epsilon = (N-1)/N$ and $R_{max} = N$. This implies that noise variance is reduced by a factor of N in a CML as compared to noise variance in a single isolated map. Notice that $\epsilon = (N-1)/N$ reduces the CML to an averaging filter that averages the different noise terms across the lattice,

$$x_n^{i+1} = (1 - \frac{N-1}{N})f[x_n^i] + \frac{(N-1)/N}{N-1}\sum_{m\neq n} f[x_m^i],$$

$$= \frac{1}{N}\sum_{m=1}^{N} f[x_m^i] = \langle f[x_m^i]\rangle. \tag{10}$$

We can extend the same approach for further iterations of the map. The analysis presented above is general and applicable to all Kaneko coupled map lattices subjected to additive Gaussian noise. In this analysis, we have assumed that noise strength is low and $\lambda_1 \neq 0$ such that the equations can be linearized and second and higher order terms can be neglected. The effect of noise color becomes evident only when higher order terms are not negligible, that is, terms containing product of noise at different iterations are not negligible. As shown by our numerical simulations in the next sections, for sufficiently low noise strength the higher order terms are indeed negligible and thus our observations are robust against the color of noise. When the higher order terms become significant, the effect of different correlations can be observed. Thus, for low noise levels, our results are in agreement with those obtained by Kia *et al.* [2015] for white noise. Typically noise levels are low and first order approximations are valid in many applications like chaos computing. Thus coupled dynamics enables robust operation in the presence of colored noise also.

3. Numerical Analysis

In this section, we present our numerical simulation results of the effects of colored noise on the evolution of sine and quadratic maps. We simulate the evolution of a single map in noise free environment for a fixed number of iterations and compare its final state with a map evolving in the presence of colored noise and that of coupled maps in noisy environment to test if they exhibit the phenomenon of reduction of noise using coupled dynamics. We consider the CML given in Eq 1 and the nodal dynamics given by quadratic map in our first example and by sine map in the second example.

The maps evolve in the presence of local noise δ^i with noise variance σ^2. For generating colored noise we use the discrete analogue of the OU process with correlation parameter Γ ($|\Gamma| < 1$) such that $|\Gamma| = e^{-1/\tau_c}$ where τ_c is the correlation time [Neiman, Anishchenko & Kurths, 1994]. Thus higher value of $|\Gamma|$ signifies higher color in the noise. The noise δ^{i+1} at $i+1$ iteration is given by

$$\delta^{i+1} = \Gamma\delta^i + (1-\Gamma^2)^{1/2}\sigma\eta^{i+1}, \tag{11}$$

where η^i is white noise of unit variance.

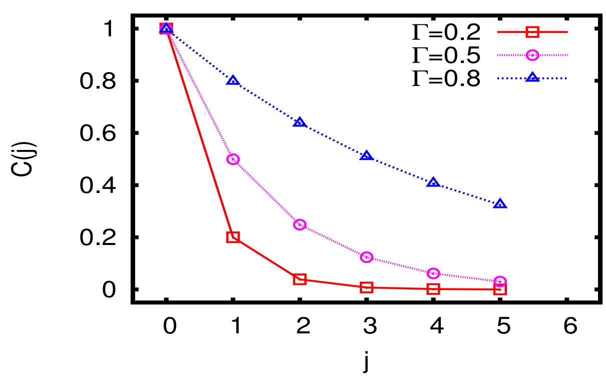

Fig. 1. Autocorrelation function for noise streams of length six for different values of correlation parameter Γ.

To measure the correlations in the noise, we generated independent noise streams of fixed length and calculated the autocorrelation function

$$C(j) = \frac{[(\delta^i - \mu)(\delta^{i+j} - \mu)]}{\sigma^2}, \tag{12}$$

where δ^i is the noise value at iteration i and μ is the mean of all noise values which is zero in our case. We averaged this quantity for 10^5 different realizations of the noise streams. In Fig. 1 we show the autocorrelation of the noise stream of length six. A higher value of $C(j)$ would imply larger dependence of noise values at $(i+j)^{th}$ iteration on noise values at i^{th} iteration. Thus $C(j) = 1$ for $j > 0$ means that noise values are same at iterations j and $i+j$ whereas $C(j) = 0$ means complete independence of noise values at different iterations as is the case for white noise.

In the next subsections, we discuss our results for the specific forms of $f[x]$, that is, the quadratic map and the sine map.

3.1. *Quadratic Map*

We consider the CML given in Eq 1 where the dynamics of each node of the lattice is given by the quadratic map

$$x^{i+1} = 1 - \lambda \left(x^i\right)^2 . \tag{13}$$

This map is chaotic in the interval $x^i \in [-1, 1]$ for $\lambda = 2$. Further, the nodes are coupled with each other with optimum coupling strength, that is, $\epsilon = (N-1)/N$. We evolve this system in the presence of local independent noise with variance $\sigma^2 = 10^{-6}$.

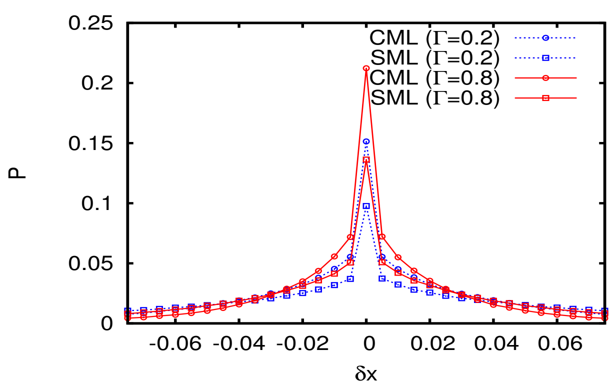

Fig. 2. The probability distributions P of noise deviations after six iterations δx for CML of size $N = 3$ evolving in the presence of noise of different correlations. The deviations are much less for coupled maps (CML) than for single maps (SML).

Similar to the procedure described in Section 2, we evolve this system for fixed number of iterations and then calculate the deviation of this system from that of noise free evolution of a single map. We repeat this procedure 10^5 times, starting with a random initial condition each time and plot the distribution of these deviations in Fig. 2 for $N = 3$. It can be seen from the figure that for fixed Γ the deviations are larger for a single map (SML) as compared to that of CML.

In Fig. 2 we also observe that the deviations are lesser for higher value of correlation parameter. This observation is true only when the deviations are averaged over random initial conditions as we also observed the deviation to be lower for lower Γ for some initial conditions. To understand the role of initial conditions, we plot the variance of deviations in a single map for 10^4 simulations for each initial condition in the interval $x^i \in [-1, 1]$ in Fig. 3.

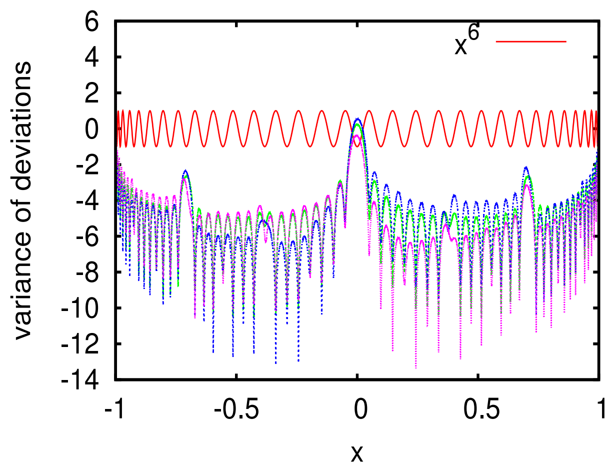

Fig. 3. The variance of deviations (logscale) of 10^4 simulations for each point $x^i \in [-1, 1]$ caused by OU noise of variance $\sigma^2 = 10^{-6}$ in a quadratic map in the chaotic region for $\Gamma = -0.8$(blue), $\Gamma = 0$(green), $\Gamma = 0.8$(pink). The top region plots the $i = 6$ iteration x^6 of the quadratic map.

We observe that different initial conditions have different sensitivity to noise and the correlations in the noise. We also observe that variance of deviations is symmetrical about $x = 0$ for $\Gamma = 0$ but the symmetry breaks for nonzero Γ and in general positive Γ results in lower deviations for $x > 0$ and vice versa though there are some exceptions to this generic trend. These differences in variance due to noise color can be observed even at second iteration. The observed differences arise from higher order terms containing multiplications of noise terms at different iterations. For example, if we write the equation of a single map at second iteration by including the second order terms also, the term containing the product of noise terms at first and second iteration is $4\lambda^2\delta_0\delta_1 x_0$ and thus this terms includes the effect of presence of correlations in the noise. Similarly, for higher iterations, the terms containing product of noise at different iterations are also multiplied by the initial condition of the map x_0 and so we observe a complex pattern as seen in

Fig. 3.

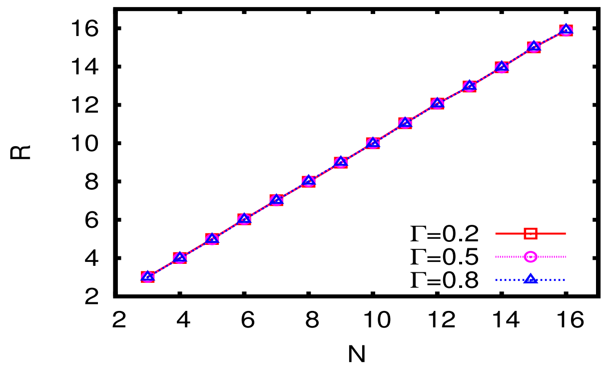

Fig. 4. Six-step noise robustness R versus lattice size N of a globally coupled CML of optimally coupled chaotic quadratic maps in the presence of noise with variance $\sigma^2 = 10^{-6}$.

To understand the effect of different number of nodes in the coupled array, we start with same initial condition for all nodes and calculate the deviations of CML after six iterations from noise free evolution for 10^5 different noise realizations. Then we calculate the variance (σ^2_{CML}) of these deviations. We also calculate the deviations of an isolated map evolving in the presence of noise from that of noise free evolution and find the variance of these deviations (σ^2_{SML}). We compute the Eq. 8 noise robustness R and average it over 80 different initial conditions. We calculated this noise robustness for varying number of nodes in the CML and for noise of different correlations as shown in Fig. 4. We used $\sigma^2 = 10^{-6}$ in these simulations and performed a consistency check for other noise intensities also. It is clear that noise robustness scales with the number of nodes in the CML for all three levels of correlations in the noise.

Even small differences are amplified exponentially in a chaotic map and after few iterations, the deviations due to noise become large enough to push the map out of basin of attraction and it starts diverging. Thus this noise robustness can be obtained only for limited number of iterations. In our simulations we observed noise robustness up to iteration six for the quadratic map evolving in noise of variance $\sigma^2 = 10^{-6}$. If noise strength is decreased, we can observe the noise robustness for higher iterations also. If the map is in periodic region, we can observe noise robustness for higher iterations also. We set $\lambda = 1.2$ in Eq. 13 and the quadratic map is in periodic regime now. The results are shown in Fig. 5 for $x_0 = 0.7$ and it can be seen that R stays 3 even for higher iteration numbers as well.

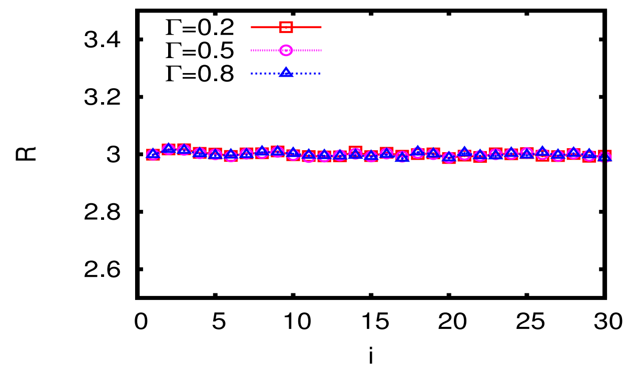

Fig. 5. Noise robustness R versus iteration number i for three coupled periodic quadratic maps ($\lambda = 1.2$).

3.2. *Generality*

To test the generality of our results, we verified the above observations for other maps and network configurations. Here we report the summary of results for sine map and nearest neighbor networks. We examined the nearest neighbor coupling scheme in which nodes are placed on a ring topology and each node is connected to its nearest neighbors only as given by Eq. 14.

$$x_n^{i+1} = (1-\epsilon) f\left[x_n^i\right] + \frac{\epsilon}{2}\left(f\left[x_{n-1}^i\right] + f\left[x_{n+1}^i\right]\right). \tag{14}$$

For three nodes, both nearest neighbor coupling and global coupling result in same configuration. Thus we performed the simulations for five nodes on a ring and found that noise can diffuse throughout the lattice resulting in higher noise robustness and it approaches that of an averaging filter for optimum number of iterations. Noise of different correlations diffused at different rates but at the optimum number of iterations, it was same for all irrespective of the correlations present in the noise.

We also studied the sine map given by

$$x^{i+1} = r \sin\left[\pi x^i\right]. \tag{15}$$

It shows chaotic behavior for $r = 1$. It has an infinite basin of attraction and remains bounded for all initial conditions at all iterations. We examined the noise robustness over $r \in (0, 1]$ and found $R = N$ for small number of iterations irrespective of the correlations in the noise. If the map is in periodic region then noise robustness of N can be observed for higher iterations also. In the chaotic regime $R = N$ for first few iterations but then decreases to 1 as after few iterations the noise spreads over the entire attractor and the deviations of the noisy map saturate. Because of the nature of chaotic dynamics we cannot permanently decrease noise deviations no matter which noise reduction technique we use. One of the main differences between linear dynamics and chaotic dynamics is that in linear dynamics, if we divide the amount of noise by two, the long term future deviations of the system from noise free evolution will be divided by two as well. But in chaotic system this is not true. This means that if we divide the amount of additive noise by two, after long enough evolution of the chaotic system we will completely lose the track of the system state and it can be anywhere in the chaotic attractor. As a result, there will be no difference between long term noise deviations in a chaotic system for different additive noise variances. However, in a chaotic system we can control and mitigate short term growth of the noise in the system as demonstrated above.

4. Coupling induced noise reduction in Chaos Computing

It was shown by Kia *et al.* [2014] that computation and noise robustness can be implemented at the dynamics level by using coupled chaotic dynamics. Computation can be realized by implementing functions in the orbits of a dynamical system [Sinha & Ditto, 1998; Kia *et al.*, 2011]. The inputs are encoded in form of initial conditions of the dynamical system which then evolves with time and after a specified time, the outputs are decoded from the state of the system at that time. The function to be realized is determined by control inputs.

Here we consider a simple example of a chaos computing system. Let us assume that we have two binary data inputs d_1 and d_2 and six digital control inputs $c_1, c_2, ...c_6$. These control and data inputs are used to construct the initial condition to be fed into the dynamical system. We construct binary encoding scheme such that the input is given by

$$I = \frac{2^7 d_1 + 2^6 d_2 + 2^5 c_1 + 2^4 c_2 + 2^3 c_3 + 2^2 c_4 + 2^1 c_5 + 2^0 c_6}{2^8} \tag{16}$$

Now let us assume that the dynamical system to be used for computation is the quadratic map given in Eq. 13. To encode the input I in an initial condition x^0 of the map, we use the simple linear scaling given by

$$x^0 = E[I] = 2I - 1. \tag{17}$$

Different number of iterations can be used for computing and the optimal iteration number is found such that it is large enough to let nearby orbits diverge, but not so large that noise effects become dominant. In our simulations we found iteration number $i = 6$ as an optimal value. We evolve the map for six iterations and then decode the output from the final state of the map using the simple decoding scheme

$$y = D[x^6] = \begin{cases} 0 & x^6 \le 0 \\ 1 & x^6 > 0, \end{cases}$$

where x^6 denotes the sixth iteration of the map starting with initial condition x^0.

Different combinations of d_1 and d_2 result in four possible inputs $d_1 d_2$ and sixteen different functions can be realized. The truth table of these functions is given in Table 1. The function to be implemented can be selected by choice of an appropriate control input. As we are using six different control inputs, these result in 64 possible controls using the binary encoding $2^5c_1+2^4c_2+2^3c_3+2^2c_4+2^1c_5+2^0c_6$. Thus there is redundancy in controls and more than one control can be used to implement same function. The controls that we used for realizing a particular function are given in the last column of Table 1.

Table 1. Truth tables and instruction set for chaos computation

Function		Output	$[d_1 d_2]$		Control
	[11]	[10]	[01]	[00]	
0	0	0	0	0	15
1	0	0	0	1	26
2	0	0	1	0	58
3	0	0	1	1	35
4	0	1	0	0	6
5	0	1	0	1	17
6	0	1	1	0	55
7	0	1	1	1	7
8	1	0	0	0	38
9	1	0	0	1	51
10	1	0	1	0	47
11	1	0	1	1	12
12	1	1	0	0	29
13	1	1	0	1	4
14	1	1	1	0	57
15	1	1	1	1	32

To examine whether coupling can enhance the robustness of chaos computing in the presence of colored noise, we implemented the above scheme with a single noisy map as well as three coupled maps. To implement a function, we initialized the nodes with the initial condition obtained by encoding of control inputs $c_1c_2...c_6$ specific to that function and data inputs d_1d_2. We iterated the map for six steps in the presence of noise of variance σ^2 and decoded the output from the final state of the map. If the output matched with the expected output corresponding to those data inputs for the given function N, it was counted as success. If the output did not match the expected output even for a single input set, it was counted as an error. We fixed a specified error rate that is acceptable and found out the maximum noise strength up to which the errors turn out to be less than the specified error rate. We call this noise strength the critical noise strength σ_c^2.

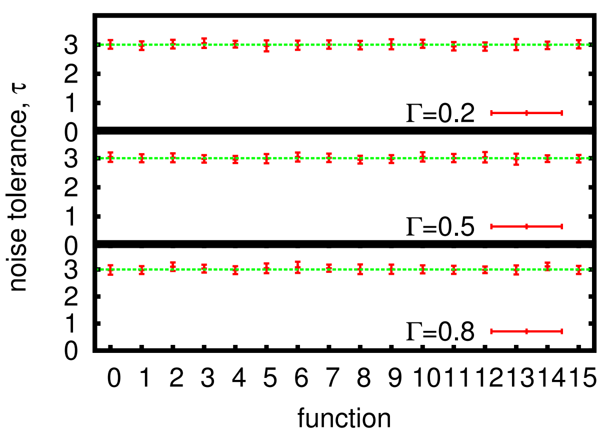

Fig. 6. Noise tolerance of various functions in chaos computing system using three coupled elements. The critical noise strength up to which errors are less than the given error rate increases three times when three coupled elements are used.

In our simulations, we fixed the error rate at 20 errors in 5000 simulations or four errors in one thousand. We started with a noise free case and kept increasing the noise strength until the number of errors became more than 20. We calculated the critical noise variance for the coupled maps σ_C^2 with optimal coupling strength as well as for a single map σ_S^2. We call the ratio of the these variances as the noise tolerance, τ.

$$\tau = \frac{\sigma_C^2}{\sigma_S^2} \tag{18}$$

We repeated the above steps 25 times. The average of these 25 values of noise tolerance along with error bars is plotted in Fig. 6 for all the 16 functions. It is clear from Fig. 6 that optimally coupled maps can mitigate the noise and same error rate can be achieved at much higher noise strengths.

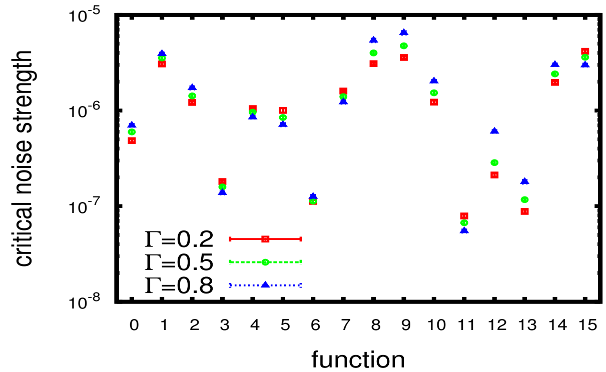

Fig. 7. Maximum noise strength up to which errors rate is less than 0.4% in chaos computing using three coupled elements for noise of different colors. The error bars of variances are less than the size of the points themselves.

The critical noise strength for noise with different correlations for an error rate of 0.4% is shown in Fig. 7. It can be observed that different functions have different sensitivity to noise and its color. Some functions like $1, 8, 9$ are more robust against noise and the critical noise strength for such functions is higher as compared to other functions. The critical noise strength also varies with the color of noise. For some functions, the presence of positive correlations in noise enhances the noise robustness, for example, functions $0, 1, 2, 6, 8, 9, 10, 12, 13, 14$ achieve same error rates at higher critical noise strength when the noise is positively correlated. Depending on the value of data input and control, each function computes the evolution of 4 different initial conditions. The initial condition least robust against noise will then determine the critical noise strength for that function. As we observed in Fig. 3 that different initial conditions have different robustness against noise and the robustness also depends on the color of the noise, the same behavior is reflected in the critical noise strength for different functions. Importantly, though the critical noise strength varies from function to function and with color of noise, we observe that the the noise tolerance is always $\tau = 3$ regardless of noise color and its effects.

5. $1/f^{\alpha}$ Noise

Another common instance of colored noise encountered in many physical systems is the $1/f^{\alpha}$ noise where α represents the scaling of the noise's power spectrum. Whereas the OU noise is stationary and Markovian, $1/f^{\alpha}$ noise is non-stationary and non-Markovian. For $\alpha = 0$, power is uniformly distributed over all frequencies and the noise is called "white noise". For non-zero α, power is unevenly distributed at various frequencies and thus the noise is called as "colored noise". Two common instances of such colored noise are "pink noise" for $\alpha = 1$ and "brown noise" for $\alpha = 2$.

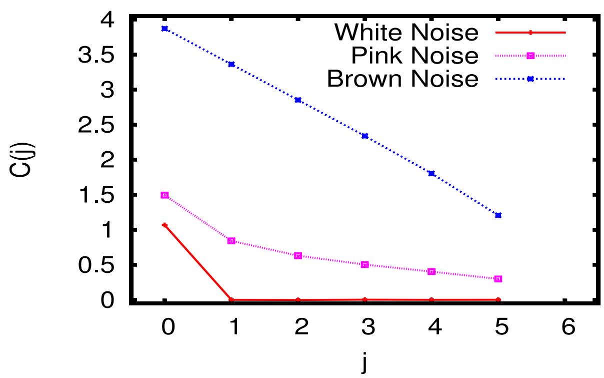

Fig. 8. Autocorrelation function for noise streams of length six for different values of noise color α.

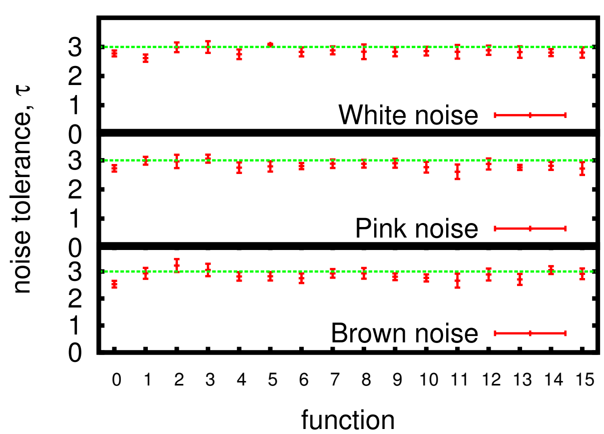

Fig. 9. Noise tolerance of various functions in chaos computing system using three coupled elements. The critical noise strength up to which errors are less than the given error rate increases three times when three coupled elements are used.

We considered three different scenarios in our simulations with varying noise correlations, (i) white noise or uncorrelated noise, (ii) pink noise and (iii) brown noise. We followed Kasdin's [1995] method for generating colored noise with different α. The autocorrelation function for the noise generated is shown in Fig. 8. As $1/f^{\alpha}$ noise is non stationary, so the variance of noise increases with time. Our extensive numerical simulations yielded the qualitative similar behavior in all the above examined cases. We obtained noise robustness of $R = N$ for coupled array containing N elements. While periodic systems exhibited noise robustness for many iterations, the chaotic ones were found to be robust for few iterations. For the sake of brevity, here we present our sample results for noise tolerance of chaos computing obtained by coupling three chaotic computing elements in the presence of $1/f^{\alpha}$ noise. We can see in Fig. 9 that noise tolerance $\tau = 3$ for all the functions.

6. Conclusions

We have studied the effect of $1/f^{\alpha}$ and OU noise on the evolution of maps and found that deviations of a map evolving in the presence of noise from that of the noise free map depend on the color of the noise. Thus noise with different correlations result in different amount of deviations. We have demonstrated that the phenomenon of reduction of local noise using coupled dynamics works irrespective of the color of the noise. The reduction occurs by a factor equal to the number of nodes used for coupling. Additionally, this phenomenon is not limited to chaotic systems and can be observed for dynamical systems in periodic regime also. The practical relevance of this averaging filter effect of coupled dynamics has been demonstrated in chaos computing in the presence of colored noise where this mechanism enabled robust operations at higher noise levels as compared to single chaotic elements.

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